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Essential Properties Related to Short-Time Fractional Fourier Transform

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Abstract. *We start by defining the short-time fractional Fourier transform in this paper, which is a natural generalization of the fractional Fourier transform. We then investigate its essential properties and explore an uncertainty principle related to this proposed transformation.*
Keywords: fractional Fourier transform, short-time fractional Fourier transform, uncertainty principle

1. Introduction

It is well known that the fractional Fourier transform (FrFT) is a generalized of the usual Fourier transform (FT), has several applications in optics, signal processing, and quantum mechanics [1,2,3,4,5,6,7]. The fractional Fourier transform depends on a parameter ϑ and can be considered as a counterclockwise rotation of the function to arbitrary angles in the spatial frequency domain. Since the FrFT is an extension of the FT, we may transfer some properties of the FT to the FrFT domain like shifting, scaling, modulation, convolution, and inequalities with some changes [8,9,10].

On the other hand, the fractional Fourier domain-frequency contents, which are needed in some applications, are not found by FrFt. This problem can be solved using short-time fractional Fourier transform (STFrFT) [11]. A crucial time–frequency analysis tool is the STFrFT [12]. It has been extensively applied in numerous signal processing domains including speech and acoustics.

The uncertainty principle (UP) is a key concept in signal processing and quantum mechanics. In many changes, one of the well-liked outcomes is likewise the uncertainty principle. A number of works, including [13,14], have recently addressed

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the uncertainty principles associated with the conventional short-time Fourier transform. The STFrFT, which may be thought of as one rotation of the conventional short-time Fourier transform, was proposed by the authors in [15]. The authors of [16] conclude that there is a connection between the FrFT parameter and the uncertainty principle for the STFrFT by reducing a fundamental inequality. Nevertheless, there are currently no publications that include the crucial characteristics and underlying principles of uncertainty for the short-time fractional Fourier transform (STFrFT).

This paper's remaining sections are arranged as follows. The fundamentals of the fractional Fourier transform (FrFT) are presented in Section 2, along with a summary of several results that are crucial to understanding the conclusions drawn in this work. The definition of the short-time fractional Fourier transform (STFrFT) is presented in Section 3. This section also includes the derivation of certain fundamental properties, such as Moyal's and the inversion formula. The attributes will yield significant benefits for the advancement of our primary findings. We also investigate a principle of uncertainty associated with this suggested transformation. Lastly, we present our conclusion in Section 4.

2. Basic Facts on Fractional Fourier Transform

In this part, we mainly review the basic notations on the fractional Fourier transform (FrFT). We start with the well known definition below.

For $1 \leq p \leq \infty$, we use the notation $L^s(\mathbb{R})$ to represent the space of measurable functions on real numbers $\mathbb{R}$ with norm

$$\|h\|_{L^s(\mathbb{R})} = \left(\int_{\mathbb{R}} |h(x)|^s \, dt\right)^{1/s} < \infty, \quad 1 \leq s < \infty. \tag{2.1}$$

In particular, when $s = \infty$ we obtain $L^\infty(\mathbb{R})$-norm

$$\|h\|_{L^\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{x \in \mathbb{R}} |h(x)|. \tag{2.2}$$

Especially, if $h \in L^\infty(\mathbb{R})$ is continuous then

$$\|h\|_{L^\infty(\mathbb{R})} = \sup_{x \in \mathbb{R}} |h(x)|. \tag{2.3}$$

The usual inner product of $L^2(\mathbb{R})$ is then defined as

$$\langle h, \phi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} h(x) \overline{\phi(x)} \, dx. \tag{2.4}$$

where $\overline{\phi(x)}$ is complex conjugate of $\phi(x)$.

We remind that the Fourier transform of a function $h \in L^2(\mathbb{R})$ is defined by

$$\mathcal{F}\{h\}(\psi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(\psi) \, e^{-ix\psi} dx. \tag{2.5}$$

Definition 2.1. For any function $h \in L^2(\mathbb{R})$, the fractional Fourier transform (FrFT) with parameter ϑ is described by the formula

$$\mathcal{F}_\vartheta\{h\}(\psi) = \int_{\mathbb{R}} h(x) K_\vartheta(\psi, x)\, dx, \tag{2.6}$$

where the kernel transform $K_\vartheta(\psi, x)$ is defined by

$$K_\vartheta(\psi, x) = \begin{cases} C_\vartheta e^{i(|x|^2+|\psi|^2)\frac{\cot\vartheta}{2} - it\cdot\psi\csc\vartheta}, & \vartheta \neq n\pi \\ \delta(x-\psi), & \vartheta = 2n\pi \\ \delta(x+\psi), & \vartheta = (2n+1)\pi, n \in \mathbb{Z}, \end{cases} \tag{2.7}$$

for δ is Dirac delta function. Here

$$C_\vartheta = \sqrt{\frac{1 - i\cot\vartheta}{2\pi}}, \quad \overline{C_\vartheta} = \sqrt{\frac{1 + i\cot\vartheta}{2\pi}}, \tag{2.8}$$

and it is evident that

$$|C_\vartheta| = \frac{1}{\sqrt{2\pi\sin\vartheta}}. \tag{2.9}$$

It is straightforward to verify that the FrFT kernel fulfills the following basic properties.

$$\overline{K_\vartheta(\psi, x)} = K_{-\vartheta}(\psi, x), \tag{2.10}$$

and

$$\int_{\mathbb{R}} K_\vartheta(\psi, x)\overline{K_\vartheta(\psi', t)}\, dt = \delta(\psi - \psi'), \tag{2.11}$$

where $\overline{K_\vartheta(\psi, x)}$ is the complex conjugate of $K_\vartheta(\psi, x)$.

Definition 2.2. For any function $h \in L^2(\mathbb{R})$ with $\mathcal{F}_\vartheta\{h\} \in L^2(\mathbb{R})$. The inversion formula of the fractional Fourier transform (FrFT) of the function h is described as

$$\begin{aligned} h(x) &= \mathcal{F}_\vartheta^{-1}\left[\mathcal{F}_\vartheta\{h\}\right](x) \\ &= \int_{\mathbb{R}} \mathcal{F}_\vartheta\{h\}(\psi)\overline{K_\vartheta(\psi, x)}\, d\psi \\ &= \int_{\mathbb{R}} \mathcal{F}_\vartheta\{h\}(\psi)\overline{C_\vartheta} e^{-i(|x|^2+|\psi|^2)\frac{\cot\vartheta}{2} + ix\cdot\psi\csc\vartheta}\, d\psi. \end{aligned} \tag{2.12}$$

Equation (2.12) tell us to recover the original function from its FrFT.

Lemma 2.3 (FrFT Parseval [17]). For two functions $h, \phi \in L^2(\mathbb{R})$, it holds

$$\langle h, \phi\rangle_{L^2(\mathbb{R})} = \langle \mathcal{F}_\vartheta\{h\}, \mathcal{F}_\vartheta\{\phi\}\rangle_{L^2(\mathbb{R})}, \tag{2.13}$$

and

$$\|h\|^2_{L^2(\mathbb{R})} = \|\mathcal{F}_\vartheta\{h\}\|^2_{L^2(\mathbb{R})}. \tag{2.14}$$

3. Short-Time Fractional Fourier Transform and Its Essential Properties

This part begins by defining the short-time fractional Fourier transform (STFrFT) which is a generalized of the classical fractional Fourier transform. We then collect its important properties which will be needed later.

Definition 3.1. For any function $\phi \in L^{\infty}(\mathbb{R})$ and $h \in L^2(\mathbb{R})$, the short-time fractional Fourier transform (STFrFT) with respect to ϕ is described by

$$\begin{aligned} \mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi) &= C_{\vartheta}\int_{\mathbb{R}} h(x)\overline{\phi(x-t)}e^{i\left(\left(\frac{x^2+\psi^2}{2}\right)\cot\vartheta - ix\psi\csc\vartheta\right)}dx \\ &= \int_{\mathbb{R}} h(x)\overline{\phi(x-t)}K_{\vartheta}(x,\psi)dx. \end{aligned} \tag{3.1}$$

For a fixed t, equation (3.1) may be expressed in the form

$$\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi) = \mathcal{F}_{\vartheta}\{\breve{A}_{f,\overline{\phi}}\}(\psi), \tag{3.2}$$

where

$$\breve{A}_{f,\overline{\phi}}(x,t) = h(x)\overline{\phi(x-t)}. \tag{3.3}$$

Remark 3.2. It should be observed that when $\vartheta = \frac{\pi}{2}$, equation (3.1) becomes

$$\begin{aligned} \mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \breve{A}_{h,\phi}(x,t)e^{-ix\psi}dx \\ &= \mathcal{F}_{\vartheta}\{\breve{A}_{h,\phi}\}(\psi). \end{aligned} \tag{3.4}$$

The fundamental relationship of the STFrFT with classical FT is given by

$$\begin{aligned} C_{\vartheta}^{-1}\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)e^{-i\psi^2\frac{\cot\vartheta}{2}} &= \int_{\mathbb{R}} \breve{A}_{h,\overline{\phi}}^{\vartheta}(x,t)e^{-ix\psi\csc\vartheta}dx \\ &= \mathcal{F}\left\{\breve{A}_{h,\overline{\phi}}^{\vartheta}(x,t)\right\}(\psi\csc\vartheta), \end{aligned} \tag{3.5}$$

where

$$\breve{A}_{h,\phi}^{\vartheta}(x,t) = h(x)\phi(x-t)e^{\frac{it^2\cot\vartheta}{2}}. \tag{3.6}$$

Some of the essential properties of the STFrFT are collected in the following theorems.

Theorem 3.3 (Moyal's formula). Let $h,\phi \in L^2(\mathbb{R})$, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^2 d\psi dt = \|f\|_{L^2(\mathbb{R})}^2\|\phi\|_{L^2(\mathbb{R})}^2. \tag{3.7}$$

Proof. In view of Parseval's formula for the FrFT in equation (2.14) and relation

(3.4), we have

$$\begin{aligned}\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^2 d\psi &= \int_{\mathbb{R}} \left|\mathcal{F}\left\{\breve{A}_{h,\phi}\right\}(\psi)\right|^2 d\psi \\ &= \int_{\mathbb{R}} \left|\breve{A}_{h,\phi}(\psi)\right|^2 dx \\ &= \int_{\mathbb{R}} \breve{A}_{h,\overline{\phi}}(\psi)\overline{\breve{A}_{h,\overline{\phi}}(\psi)} d\psi \\ &= \int_{\mathbb{R}} h(x)\overline{\phi(x-t)}\overline{h(x)\overline{\phi(x-t)}} dx \\ &= \int_{\mathbb{R}} h(x)\overline{h(x)\phi(x-t)}\phi(x-t) dx. \end{aligned} \tag{3.8}$$

We integrate the both sides of equtaion (3.8) with respect to dt and get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^2 d\psi dt = \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\overline{h(x)\phi(x-t)}\phi(x-t) dx dt. \tag{3.9}$$

Prforming the change of variables $x - t = u$, we infer that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^2 d\psi dt = \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\overline{h(x)\phi(u)}\phi(u) dx du. \tag{3.10}$$

Using Fubini's theorem on the right side of equation (3.10), we further get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^2 d\psi dt = \int_{\mathbb{R}} |h(x)|^2 dx \int_{\mathbb{R}} |\phi(u)|^2 du, \tag{3.11}$$

or,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^2 d\psi dt = \|h\|^2_{L^2(\mathbb{R})}\|\phi\|^2_{L^2(\mathbb{R})},$$

which completes the proof. □

Theorem 3.4 (Inversion formula). Let $h, \phi \in L^2(\mathbb{R})$, then inversion formula of the STFrFT is given by

$$h(x) = \frac{1}{\overline{\phi(0)}} \int_{\mathbb{R}} \mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi) K_{-\vartheta}(x,\psi) dx. \tag{3.12}$$

Proof. It follows with relation (3.1) that

$$\begin{aligned}\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi) &= C_\vartheta \int_{\mathbb{R}} h(x)\overline{\phi(x-t)} e^{i\left(\left(\frac{x^2+\psi^2}{2}\right)\cot\vartheta - ix\psi\csc\vartheta\right)} dx \\ &= \int_{\mathbb{R}} \breve{A}_{f,\overline{\phi}}(x,t) K_\vartheta(x,\psi) dx, \end{aligned} \tag{3.13}$$

where $\breve{A}_{f,\overline{\phi}}(x,t)$ is defined by equation (3.3). Using inversion of FrFT in (2.12), then

$$\breve{A}_{f,\overline{\phi}}(x,t) = \int_{\mathbb{R}} \mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi) K_{-\vartheta}(x,\psi) dx. \tag{3.14}$$

Equation equation (3.14) above can be rewriten in the form

$$h(x)\overline{\phi(x-t)} = \int_{\mathbb{R}} \mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi) K_{-\vartheta}(x,\psi) dx. \tag{3.15}$$

Choosing $x = t$ in equation (3.15), we obtain

$$h(x)\overline{\phi(0)} = \int_{\mathbb{R}} \mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)K_{-\vartheta}(x,\psi)dx. \tag{3.16}$$

It means that

$$h(x) = \frac{1}{\overline{\phi(0)}} \int_{\mathbb{R}} \mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)K_{-\vartheta}(x,\psi)dx.$$

Thus, the desired result is obtained. □

Next, we will build Sharp Housdorff-Young inequality for the short-time fractional Fourier transform (STFrFT), which generalizes Sharp Hausdorff-Young inequality for the FT in StFrFT domains.

Theorem 3.5. Let $p(1 \leq r \leq 2)$ and q such that $\frac{1}{p}+\frac{1}{q} = 1$, then for any $h, \phi \in L^p(\mathbb{R})$, we have

$$\left\|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right\|_{L^q(\mathbb{R}\times\mathbb{R})} \leq \frac{|\sin\vartheta|^{\frac{1}{q}-\frac{1}{2}}p^{\frac{1}{2p}}}{\sqrt{2\pi}q^{\frac{1}{2q}}}\|h\|_{L^p(\mathbb{R})}\|\phi\|_{L^p(\mathbb{R})} \tag{3.17}$$

Proof. With aid of Sharp Housdorff-Young inequality for the FT, we find that

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{h(x)\}(\psi)|^q d\psi\right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}} |h(x)|^p dx\right)^{\frac{1}{p}}. \tag{3.18}$$

We replace $h(x)$ by $\breve{A}_{h,\phi}^{\vartheta}(x,t)$ by (3.6) into both sides of equation (3.18) above, we get

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{\breve{A}_{h,\phi}^{\vartheta}\}(\psi)|^q d\psi\right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}} |\breve{A}_{h,\phi}^{\vartheta}(x,t)|^p dx\right)^{\frac{1}{p}}. \tag{3.19}$$

We setting $\psi = \psi\csc\vartheta$ in equation (3.19) above, we obtain that

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{\breve{A}_{h,\phi}^{\vartheta}\}(\psi\csc\vartheta)|^q d(\psi\csc\vartheta)\right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}} |\breve{A}_{h,\phi}^{\vartheta}(x,t)|^p dx\right)^{\frac{1}{p}}. \tag{3.20}$$

Applying (3.5) to left hand side of eqution (3.20), we find

$$\left((2\pi)^{\frac{q}{2}}|\sin\vartheta|^{\frac{q}{2}-1}\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^q d\psi\right)^{\frac{1}{q}} \leq \frac{p^{\frac{1}{2p}}}{q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}} |h(x)\overline{\phi(x-t)}|^p dx\right)^{\frac{1}{p}}. \tag{3.21}$$

Hence,

$$\left(\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^q d\psi\right)^{\frac{1}{q}} \leq \frac{|\sin\vartheta|^{\frac{1}{q}-\frac{1}{2}}p^{\frac{1}{2p}}}{\sqrt{2\pi}q^{\frac{1}{2q}}} \left(\int_{\mathbb{R}} |h(x)|^p|\overline{\phi(x-t)}|^p dx\right)^{\frac{1}{p}}, \tag{3.22}$$

and

$$\int_{\mathbb{R}} \left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^q d\psi \leq \left(\frac{|\sin\vartheta|^{\frac{1}{q}-\frac{1}{2}}p^{\frac{1}{2p}}}{\sqrt{2\pi}q^{\frac{1}{2q}}}\right)^q \left(\int_{\mathbb{R}} |h(x)|^p|\overline{\phi(x-t)}|^p dx\right)^{\frac{q}{p}}. \tag{3.23}$$

Integrating both sides equation (3.23) with respect to dt we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^{q}d\psi dt \leq \left(\frac{|\sin\vartheta|^{\frac{1}{q}-\frac{1}{2}}p^{\frac{1}{2p}}}{\sqrt{2\pi}q^{\frac{1}{2q}}}\right)^{q}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|h(x)|^{p}|\overline{\phi(x-t)}|^{p}dxdt\right)^{\frac{q}{p}}. \tag{3.24}$$

Equation (3.24) may be expressed as

$$\begin{aligned}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right|^{q}d\psi dt\right)^{\frac{1}{q}} &\leq \frac{|\sin\vartheta|^{\frac{1}{q}-\frac{1}{2}}p^{\frac{1}{2p}}}{\sqrt{2\pi}q^{\frac{1}{2q}}}\left(\int_{\mathbb{R}}|h(x)|^{p}dx\int_{\mathbb{R}}|\overline{\phi(x-t)}|^{p}dt\right)^{\frac{1}{p}}\\ &= \frac{|\sin\vartheta|^{\frac{1}{q}-\frac{1}{2}}p^{\frac{1}{2p}}}{\sqrt{2\pi}q^{\frac{1}{2q}}}\|f\|_{L^{p}(\mathbb{R})}\|\phi\|_{L^{p}(\mathbb{R})}. \end{aligned} \tag{3.25}$$

The above equation is the same as

$$\left\|\mathcal{STF}_{\phi}^{\vartheta}\{h\}(t,\psi)\right\|_{L^{q}(\mathbb{R}\times\mathbb{R})} \leq \frac{|\sin\vartheta|^{\frac{1}{q}-\frac{1}{2}}p^{\frac{1}{2p}}}{\sqrt{2\pi}q^{\frac{1}{2q}}}\|h\|_{L^{p}(\mathbb{R})}\|\phi\|_{L^{p}(\mathbb{R})}.$$

We obtained the desired result. □

4. Conclusion

In this paper, we have introduced the short-time fractional Fourier transform (STFrFT) and investigated its essential properties. We have investigated Sharp Housdorff-Young inequality related to this transformation by applying several relations and properties that have been obtained for the proposed transformation.

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